國立嘉義大學九十三學年度
光電暨固態電子研究所碩士班招生考試試題

科目：工程數學

一、For a tridiagonal matrix $\mathbf{H} = \begin{pmatrix} 0 & \gamma & 0 & 0 & \cdots \\ \gamma & 0 & \gamma & 0 & \cdots \\ 0 & \gamma & 0 & \gamma & \cdots \\ 0 & 0 & \gamma & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$, find the upper left corner element of $(E\mathbf{I} - \mathbf{H})^{-1}$? (note: $\mathbf{I}$ is the unit vector) (10%)

二、For the Pauli spin matrices $\mathbf{S}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \mathbf{S}_y = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}$ and $\mathbf{S}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, the raising and lowing spins are $\mathbf{S}^+ = \mathbf{S}_x + \mathrm{i}\mathbf{S}_x$, $\mathbf{S}^- = \mathbf{S}_x - \mathrm{i}\mathbf{S}_y$, respectively. Please calculate the commute of $[\mathbf{S}_z, \mathbf{S}^+]$ and $[\mathbf{S}^+, \mathbf{S}^-]$, where the commuter means $[A, B] \equiv AB - BA$. (15%)

三、There is an unknown form of function $f(x)$, and we want to calculate its second derivative $\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}$ at point a, if we only have $f(a) = 0.5$, $f(a+\delta x) = 0.53$ and $f(a-\delta x) = 0.52$, where δx is an infinitesimal small value, $\delta x = 0.01$. What is $\frac{\mathrm{d}^2 f(x)}{\mathrm{d}x^2}\Big|_{x=a} = ?$ (10%)

四、Solve the differential equation $y'' + 10y' + 25y = 0$, with the boundary conditions $y(0) = 3$ and $y'(1) = \mathrm{e}^{-5}$. (15%)

五、The solution of the differential equation $\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + 2\beta\frac{\mathrm{d}x}{\mathrm{d}t} + \omega_0^2 x = \cos\omega t$, $2\beta^2 < \omega_0^2$, $\omega > 0$ is given by $x(t) = \mathrm{e}^{-\beta t}(C_1\cos\omega_1 t + C_2\sin\omega_1 t) + D_1\cos\omega t + D_2\sin\omega t$, where $\omega_1^2 = \omega_0^2 - \beta^2$. The amplitude of steady state is given by $D(\omega) = \sqrt{D_1^2 + D_2^2}$. Find the maximum of the amplitude $D_{\max} = D(\omega = \omega_{\mathrm{R}})$ and the resonant frequency ω_{R}. (10%)

六、The system of an undamped pendulum is described by $\frac{\mathrm{d}\theta}{\mathrm{d}t} = \omega$, $\frac{\mathrm{d}\omega}{\mathrm{d}t} = -k\sin\theta$, where $k = g/L$. In the sense of Liapunov, the nature of the critical point is classified as (1) asymptotically stable (stable and attractive), (2) stable but not asymptotically stable (stable but not attractive), or (3) unstable; and the type can be described as (a) a node, (b) a saddle point, (c) a centre, or (d) a spiral point. Classify the critical points $(\theta = 0, \omega = 0)$, and $(\theta = \pi, \omega = 0)$ in the (θ, ω) phase diagram. (note: $\sin x \approx x - x^3/6 + \ldots$) (10%)

七、Consider the eigenfunction problem: $\frac{\mathrm{d}^2 y_n}{\mathrm{d}x^2} + E_n^2 y_n = 0$, $0 \le x \le 1$, $y_n(0) = 0$, $\left(\frac{\mathrm{d}y_n}{\mathrm{d}x}\right)_{x=1} = -y_n(1)$. Consider the n-th eigenvalues of E_n (for $E_1 < E_2 < \ldots < E_n$), find the corresponding eigenfunctions $y_n(x)$, and derive $\lim_{n\to\infty}\frac{E_n}{n-1}$. (note: $\tan x \approx x + \frac{x^3}{3} + \ldots$ for $|x| < \pi/2$) (15%)

八、The orthogonality relationship for Bessel function of the order 1/2 is given by $\int_0^k r \cdot J_{1/2}(\frac{j_m}{k}r) \cdot J_{1/2}(\frac{j_n}{k}r)\,\mathrm{d}r = 0$ if $m \ne n$, where $J_{1/2}(k) = 0$, and j_m, j_n are the m-th and n-th zeros of $J_{1/2}(r) = 0$. For $k = \pi$ and $f(r) = \sqrt{r}\sin r$, find A_1, A_2 in the Fourier-Bessel expansion $f(r) = \sum_{n=1}^{\infty} A_n J_{1/2}(\frac{j_n}{k}r)$. (Note: $J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\sin x$, $\int_0^\pi x\sin nx\sin x\,\mathrm{d}x = -\frac{1}{(n^2-1)^2}[2n + 2n\cos n\pi + (n^2-1)\pi\sin n\pi]$.) (15%)