國立嘉義大學九十五學年度
應用數學系碩士班招生考試試題

科目：高等微積分

說明：本試題為計算、證明題，請標明題號，同時將過程作答在「答案卷」上。

1. (a) Show that the series $\sum_{n=1}^{\infty} \frac{1}{n!}$ converges.（10%）

   (b) Show that $\int_0^1 \left(\frac{\sin x}{x^2}\right) \mathrm{d}x$ is divergent.（10%）

2. (a) Prove that $f(x) = 1/(x^2+1)$ is uniformly continuous on the real line $\Re$.（10%）

   (b) Find all real α such that $x^{\alpha}\sin(1/x)$ is uniformly continuous on the open interval $(0,1)$.（10%）

3. Show that the sequence $\{a_n\}_{n=1}^{\infty}$ converges and find its limit, where $a_1 = \sqrt{2}$ and $a_{n+1} = \sqrt{2+\sqrt{a_n}}$ for $n \geq 1$.（15%）

4. Suppose $\{f_n(x)\}$ is a sequence of continuous functions on $[0, 2]$ and satisfies $f_n(x) = \frac{1}{n+1} + \int_0^x f_n^{\,2}(t)\,\mathrm{d}t$, $x \in [0, 2]$, $\forall\, n \in \aleph$. Show that $\{f_n(x)\}$ converges uniformly to 0 on $[0, 2]$.（15%）

5. (a) Let $\{x_n\}$ be a bounded sequence of real numbers. Define $a_n = \inf\{x_k \mid k \geq n\}$. Prove or disprove that the sequence $\{a_n\}$ converges.（7%）

   (b) Suppose $A \subset \Re$ is compact and nonempty. Show that $\inf A \in A$.（8%）

6. Let A and B be nonempty subsets of $\Re^n$. Define $d(A,B) = \inf\{\|x-y\| \mid x \in A,\ y \in B\}$.

   (a) Suppose $A = \{a\}$ and B is closed. Prove that there exists $b \in B$ such that $d(A,B) = \|a-b\|$.（7%）

   (b) Suppose A is compact and B is closed. Prove that there exist $p \in A,\ q \in B$ such that $d(A,B) = \|p-q\|$.（8%）