

國立嘉義大學九十四學年度
應用數學系碩士班招生考試試題

科目：高等微積分

說明：本考試試題為計算、證明題，請標明題號，同時將過程作答在「答案卷」上。

(1~2 題每題 20 分，3~6 題每題 15 分，共 100 分)

1. (a). Evaluate $\lim_{n \rightarrow \infty} \int_0^1 \frac{e^x \sin nx}{n} dx$. (10 分)
(b). Evaluate $\lim_{n \rightarrow \infty} \int_0^1 e^{-n x^2} x^3 dx$. (10 分)
2. (a). Suppose that f is nonnegative continuous on $[a, b]$ and g is a decreasing positive continuous function on $[a, b]$. Prove that there exists a number $c \in [a, b]$ such that $\int_a^b f(x) g(x) dx = g(a) \int_a^c f(x) dx$. (10 分)
(b). Let $f: [-3, 3] \rightarrow \mathbb{R}$ be a function and let f' be continuous on $[-3, 3]$. Define $r(f) = \int_{-3}^3 f(t) dt - 6 f(0)$. Show that $|r(f)| \leq 9 \max_{t \in [-3, 3]} |f'(t)|$. (10 分)
3. Let $\{a_n\}$ be a real sequence and $\lim_{n \rightarrow \infty} a_n = a$. Show that $\lim_{n \rightarrow \infty} \frac{a_1 + a_2 + \cdots + a_n}{n} = a$. (15 分)
4. (a). Show that $F(x) = \int_0^x e^{-t^2} dt$ is a uniformly continuous function on the open interval $(0, 1)$. (8 分)
(b). Is $f(x) = x^2 - 10$ uniformly continuous on \mathbb{R} ? Justify your answer! (7 分)
5. Let $B = \left\{ x \in \mathbb{R} \mid \sin \frac{1}{x} = 0 \right\}$. Prove or disprove the following statements.
(a). B is closed in \mathbb{R} . (7 分)
(b). $\int_1^\infty \sin \frac{1}{x} dx$ is divergent. (8 分)
6. (a). Given $a_1 = -1$, $a_2 = -2$, and $a_n = 3 a_{n-1} + a_{n-2}$, $\forall n \geq 3$, find $\lim_{n \rightarrow \infty} \frac{a_{n-1}}{a_n}$. (5 分)
(b). Given $f_n(x) = 3 x^{n+1} + x^n - 5$, $x \in [0, 1]$, $\forall n \in \mathbb{N}$, find $\lim_{n \rightarrow \infty} f_n(x)$. Is the sequence $\{f_n(x)\}$ converges uniformly on $[0, 1]$? Why? (10 分)